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Research and analysis

Research review series: mathematics

Published 25 May 2021

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Introduction

Mathematics, a universal language that enables understanding of the world, is an integral part of the curriculum. Beyond the study of numbers, shapes and patterns, it also provides important tools for work in fields such as engineering, physics, architecture, medicine and business. It nurtures the development of a logical and methodical mindset, as well helping to inculcate focus and the ability to solve all manner of problems. Attainment in the subject is also the key to opening new doors to further study and employment. However, despite its importance, for many the subject remains mysterious and difficult, the preserve of those who seem to be 'naturals'. The education inspection framework (EIF) makes it clear that schools are expected to ensure that the mathematics curriculum 'helps pupils to gain enjoyment through a growing self-confidence in their ability'.[footnote 1]

This review explores the literature relating to the field of maths education. Its purpose is to identify factors that can contribute to high-quality school maths curriculums, assessment, pedagogy and systems. We will use this understanding of subject quality to examine how maths is taught in England's schools from Reception onwards. We will then publish a subject report to share what we have learned.

The purpose of this research review and the intended audience is outlined more fully in the 'Principles behind Ofsted's research reviews and subject reports'.[footnote 2]

Since there are a variety of ways that schools can construct and teach a high-quality maths curriculum, it is important to recognise that there is no singular way of achieving high-quality maths education.

In this review, we have:

- outlined the national context in relation to maths
- summarised our review of research into factors that can affect quality of education in maths
- considered curriculum progression in maths, pedagogy, assessment and the impact of school leaders' decisions on provision

The review draws on a range of sources, including our 'Education inspection framework: overview of research' and our 3 phases of curriculum research.[footnote 3]

We hope that through this work, we will contribute to raising the quality of maths education for all young people.

Ambition for all

Summary

This review identifies that, despite English pupils achieving, on average, higher attainment than pupils in many other countries, the attainment gap between low and high achievers in England is wide. Therefore, in addition to shining a light on approaches that could raise the attainment of all pupils still further, a core theme of this review is how we might prevent struggling pupils from falling further behind their peers.

Context

England performs well in mathematics compared with other countries[footnote 4] and mathematics continues to be the most popular subject to study at A level.[footnote 5] In 2004, the government carried out an inquiry into post-14 mathematics.[footnote 6] The interventions that followed this inquiry contributed to positive changes. These actions included:

- revisions to A-level mathematics specifications
- redesigning of GCSE mathematics specifications to increase their rigour and challenge
- establishing the National Centre for Excellence in the Teaching of Mathematics
- setting higher targets for teacher recruitment and creating professional development programmes for teachers

There is still more that could be done to enhance mathematics education, such as reducing the shortage of specialist mathematics teachers.[footnote 7] Additionally, the gap between the lowest and highest achievers in England is wider than the Organisation for Economic Co-operation and Development (OECD) average.[footnote 8] The average attainment gap between disadvantaged and advantaged pupils is also wide.[footnote 9] Disadvantaged pupils in England are much less likely than their advantaged peers to achieve a grade 4 at GCSE[footnote 10] or to meet the expected standards at key stages 1 and 2 or at the end of the early years foundation stage (EYFS).[footnote 11] The factors that explain the variation in quality of mathematics education in England are therefore likely to contribute to a long tail of underachievement and a wide spread of attainment level, as well as to overall success.

Recent influences in mathematics education include:

- the National Numeracy Strategy[footnote 12]
- the Mathematics Teaching Exchange[footnote 13]
- the Teaching for Mastery Programme[footnote 14] (as related to the Mathematics Teaching Exchange)

'Mastery' pedagogical approaches that have influenced English mathematics education tend to require pupils to demonstrate high levels of achievement before they are moved on to new content. Some mastery approaches place a greater emphasis on problem-solving and on deepening pupils' understanding.[footnote 15]

Cultural factors, such as the drive for and veneration of exam success in the subject, as well as an emphasis on effort over ability, may, though, exert significant influence on pupils' dispositions.[footnote 16] This, for example, may be the reason why 75% of Chinese pupils in English schools on free school meals achieved the expected standard in mathematics at key stage 2 in 2019 compared with 44% of their White British counterparts.[footnote 17]

It is also important to consider that high attainment and proficiency of older pupils may be due to historical curricular and pedagogical approaches, rather than the educational approaches of that time. Finland is a good example of an education system where success in the OECD's Programme for International Student Assessment (PISA) is thought to be the result of historical approaches.[footnote 18] Approaches that teachers and pupils are familiar with can take time to change. They may, therefore, influence pupils' educational experiences, even after an official change of curriculum or pedagogical approach.[footnote 19]

Principles underpinning the review process

This research has been informed by the evidence and principles underpinning the EIF, which include:

- scope, content and sequencing of the curriculum
- specification and ordering of component parts that make up composite skills
- the value of teachers' subject knowledge
- promotion of a range of quality interactions with pupils
- quality and pacing of instruction
- how to avoid overloading working memory
- the value of deliberate practice, interleaving and regular low-stakes testing[footnote 20]

These have become a useful lens through which to scrutinise subject-specific themes.

This review seeks to make a clear distinction between mathematics curriculum and pedagogy. We have also classified mathematics curriculum content. We have used these classifications in our review of the available literature. We have drawn forms or categories of content from disciplines in which mathematics is applied.[footnote 21] These categories are informed by the way our minds work[footnote 22] and are intended to be easy to understand.

Mathematics research tends to use a wide variety of overlapping terms. 'Fluency' is a good example. It has multiple meanings in literature:

- sometimes it refers to ease of recall and computation (which the review refers to as 'automaticity')
- sometimes it refers to conceptual knowledge

Terms also vary over time. For example, 'tables' has changed in meaning: it used to refer to number facts in all the operations that infant and junior pupils were expected to learn[footnote 23] but now it is shorthand for multiplication facts in the present day.[footnote 24] Examples of terms (and associated concepts) that have been used less and less over time include 'mechanical drill', 'syllabus' and 'recall'.

How the review classifies mathematics curriculum content

For this review, we have classified mathematical curriculum content into declarative, procedural and conditional knowledge.

Declarative knowledge is static in nature and consists of facts, formulae, concepts, principles and rules.

All content in this category can be prefaced with the sentence stem 'I know that'.

Procedural knowledge is recalled as a sequence of steps. The category includes methods, algorithms and procedures: everything from long division, ways of setting out calculations in workbooks to the familiar step-by-step approaches to solving quadratic equations.

All content in this category can be prefaced by the sentence stem 'I know how'.

Conditional knowledge gives pupils the ability to reason and solve problems. Useful combinations of declarative and procedural knowledge are transformed into strategies when pupils learn to match the problem types that they can be used for.

All content in this category can be prefaced by the sentence stem 'I know when'.

When pupils learn and use declarative, procedural and conditional knowledge, their knowledge of relationships between concepts develops over time.[footnote 25] This knowledge is classified within the ‘type 2’ sub-category of content (see table below). For example, recognition of the deep mathematical structures of problems and their connection to core strategies is the type 2 form of conditional knowledge.

Summary table of content categories considered in the review:

Category	Type 1	Type 2
Declarative ‘I know that’	Facts and formulae	Relationship between facts (conceptual understanding)
Procedural ‘I know how’	Methods	Relationship between facts, procedures and missing facts (principles/mechanisms)
Conditional ‘I know when’	Strategies	Relationship between information, strategies and missing information (reasoning)

Curriculum progression: the planned and purposeful journey to expertise

Summary

The evidence presented here supports careful consideration of sequencing and content that makes a mathematics curriculum a guarantee of long-term learning. Useful facts and efficient and accurate methods are ideally paired within a topic sequence. Strategies for solving problem types are then best taught and learned once pupils can recall and deploy facts and methods with speed and accuracy. When planning curriculum content, teachers also need to prioritise ‘forward-facing’ knowledge. This goes beyond important facts of number. It includes the mathematical methods that pupils will take with them on their journey. The ideal aim is for pupils to attain proficiency, not just collective moments of understanding, familiarity or experience. This will help pupils to develop motivation in the subject.

Selecting and sequencing core declarative, procedural and conditional knowledge

The mathematics curriculum is the product of careful selection, sequencing and linking of declarative, procedural and conditional knowledge. Pupils need to systematically acquire core mathematical facts, concepts, methods and strategies to be able to experience success when problem-solving and in order to become proficient mathematicians.[footnote 26] Opportunities to develop what are assumed to be generic skills for problem-solving, such as analysis and evaluation, should not circumvent this process.[footnote 27] Careful sequencing of content, instruction and rehearsal can also show pupils new and consistent patterns of useful information. These then form the basis of further concepts, rules and principles that pupils can store in their long-term memory.[footnote 28]

Problem-solving requires pupils to hold a line of thought. It is not easy to learn, rehearse or experience if the facts and methods that form part of a strategy for solving a problem type are unfamiliar and take up too much working memory. For example, pupils are unlikely to be able to solve an area word problem that requires them to multiply 2 lengths with different units of measurement if they do not realise that the question asks them to use a strategy to find an area. They are also unlikely to be successful if they do not know many number bonds, unit measurement facts, conversion formula or an efficient method of multiplication to automaticity. Therefore, the initial focus of any sequence of learning should be that pupils are familiar with the facts and methods that will form the strategies taught and applied later in the topic sequence.[footnote 29]

The relationship between core mathematical facts and powerful methods

Linked declarative and procedural knowledge are ideally sequenced together to reflect the reciprocal learning relationship between them. This is because:

- familiarity with the facts being used helps with learning and understanding the linked method
- familiarity with the method helps to make associated facts firm and precise in the mind

As a simple example, a pupil can better understand connections of number and the concepts of addition and quantity if they have declarative knowledge of number bonds and procedural knowledge of column addition, which both reinforce each other. In terms of curriculum sequencing, pupils are able to retain knowledge and ability to use core methods when teachers take an iterative approach to teaching and rehearsing concepts and core methods.[footnote 30]

The long-term impact of an early and thorough emphasis on core content

Acquiring new foundational knowledge takes time and effort. However, the rewards go beyond the immediate benefits of being able to recall and apply useful facts and methods.

Foundational knowledge, particularly proficiency in number, gives pupils the ability to progress through the curriculum at increasing rates later on.[footnote 31] The path of learning that begins with a diligent focus on core declarative and procedural knowledge is not a straight line, therefore, but a curve. This is a function of the curriculum's intelligent design. For example, in countries where pupils do well, pupils are able to attempt more advanced aspects of multiplication and division in Year 4 if they have been given more time on basic arithmetic in Year 1.[footnote 32] This may explain why successful curriculum approaches tend to emphasise core knowledge early on.[footnote 33]

Furthermore, if this core content has been sequenced well and pupils have learned it thoroughly, they are less likely to forget and are therefore unlikely to need to 're-learn' it later.[footnote 34] A focus on core knowledge in younger year groups can be achieved by focusing on depth over breadth, covering fewer core topics but in more detail.

Amplifying the curriculum through instruction, rehearsal and assessment plans

Successful curriculums illustrate the importance of detail, sequencing and alignment of content, instruction, rehearsal, assessment and mechanisms to continually upgrade.[footnote 35]

Textbooks, lesson plans and resources are common features of successful approaches.[footnote 36] They ensure that pupils' acquisition of content mirrors the curriculum sequence. This transforms a curriculum offer into more of a guarantee. Teachers in these systems also have more time to focus on how to bring mathematics content to life instead of redesigning sequences of content, instruction and rehearsal from scratch.[footnote 37] In education systems where the content and the sequence of content is decided more centrally, pupils can move between schools with minimal disruption to their learning.[footnote 38]

The approach outlined above is very different to a curricular offer that does not feature systems for documenting quality sequences of instruction and rehearsal and that may result in more variable rates of learning and outcomes. For example, younger pupils may achieve proficiency through more informal opportunities to learn and where teachers respond to their interests, but leaders should note that disadvantaged novice mathematicians benefit from proactive approaches that can be as simple as ensuring that they are given dedicated time to learn and rehearse mathematics every day.[footnote 39]

The advantages of these and other highly systematic approaches apply to all age groups, including Reception Year.[footnote 40] However, too frequently, pupils fall behind and disadvantaged pupils are less likely to make progress compared with their more advantaged peers. If coherent resources for planning, instruction and rehearsal of content are provided by leaders, then this risk is reduced while still giving teachers freedom to choose how to teach. Systematic teacher-led approaches, particularly in the primary key stages, lead to better attainment.[footnote 41] These then give pupils more opportunity to succeed in secondary school.

A positive attitude towards mathematics is the outcome of success in the subject

Pupils are more likely to develop a positive attitude towards mathematics if they are successful in it,[footnote 42] especially if they are aware of their success.[footnote 43] However, teachers should be wary of the temptation to invert this causal pathway by, for example, substituting fun games into lessons as a way of fostering enjoyment and motivation. This is because using games as a learning activity can lead to less learning rather than more.[footnote 44]

Some pupils become anxious about mathematics. It is not the nature of the subject but failure to acquire knowledge that is at the root of the anxiety pathway.[footnote 45] The origins of this anxiety may have even been present at the start of a pupil's academic journey.[footnote 46] However, if teachers ensure that anxious pupils acquire core mathematical knowledge and start to experience success, those pupils will begin to associate the subject with enjoyment and motivation.

This also has implications for how mistakes are viewed by pupils and teachers. Ideally, teachers and pupils should be aware of the difference between infrequent mistakes that can be learned from and consistent mistakes that lead novice mathematicians onto the anxiety pathway. These sorts of mistakes are due to weak foundational knowledge that is more likely to generate errors and misconceptions.[footnote 47] Teachers should try to put pupils on the causal pathway that leads from success to motivation by focusing on early proficiency, rather than expecting pupils to learn through making mistakes. This proficiency-first approach is likely to prevent pupils developing anxiety. For teachers of pupils who have experienced failure, frustration and the development of anxiety, rather than removing experiences where pupils might be confronted with failure (such as tests), the evidence suggests the solution lies in closing gaps so that anxious pupils can experience more understanding, accuracy and success.[footnote 48]

Based on the above, high-quality maths education may have the following features

- Successful curriculum progression is planned from the beginning of a pupil's education through focusing on core content, to develop pupils' motivation and to allow more breadth and depth later.
- The planned curriculum details the core facts, concepts, methods and strategies that give pupils the best chance of developing proficiency in the subject.

- The teaching of linked facts and methods is sequenced to take advantage of the way that knowing facts helps pupils to learn methods and vice versa.
- Sequences of learning allow pupils to access their familiarity with the facts and methods they need in order to learn strategies for solving problem types.

Curriculum sequencing: declarative knowledge

Maths facts, vocabulary and symbols at the start of the school journey

Many pupils start school with some mathematical knowledge.[footnote 49] This is not necessarily the outcome of natural ability or a different developmental pathway. Rather, it can be an indication of parental input and early exposure to the basics in mathematics in the home.[footnote 50]

Studies indicate that this early acquisition of knowledge significantly predicts later success.[footnote 51] They also suggest that much of this success is reliant on pupils knowing the code for number (for example, the Arabic numerals), rather than relying on a general sense of quantity.[footnote 52] Early lack of knowledge also predicts later struggle[footnote 53] and a later diagnosis of disability.[footnote 54] Pupils who are not able to quickly and easily recall maths facts struggle with calculations due to their working memory being overloaded.[footnote 55] For example, a child who does not know number bonds will be stuck using various forms of 'counting on' when performing simple addition. Even at key stage 3, a pupil's lack of knowledge of the basics in number will have a detrimental effect on their learning of algebra.[footnote 56] Taken together, this information points to the prioritising core declarative knowledge in mathematics from an early age to level the playing field, particularly for pupils with special educational needs.[footnote 57]

Given that many pupils who have early knowledge have been exposed to knowledge in the home, a school's decision to rely on provision of a 'maths-rich' environment must be balanced with the needs of pupils who have not had that advantage and who are less likely to choose maths activities that are provided. Many young pupils need and benefit from systematic provision of sequenced core content that becomes the building blocks of later success.[footnote 58] For example, 'more than 100 basic addition facts must become automatic before children can play around with and contemplate [different] types of problems'.[footnote 59] At-risk pupils who are systematically taught the component parts of declarative knowledge not only benefit,[footnote 60] but have the potential to take on different trajectories of learning. They are often then able to match or even exceed the attainment of their more advantaged peers.[footnote 61]

It is especially important for children to acquire proficiency with whole numbers and fractions and for working with 2- and 3-dimensional shapes in the primary phase because of how much they are used in later topics and key stages. This includes, for example, automatic recall of number facts and familiarity with the main concepts such as the associative, distributive and commutative properties.[footnote 62] Pupils also benefit from having the fundamental features of mathematics pointed out to them, such as pattern and structure, even if they are likely to intuit this information over time.[footnote 63]

A proactive approach to helping children to acquire everyday language used to describe quantity, shape and time would also benefit disadvantaged pupils, who are more likely to misunderstand instruction and activities.[footnote 64] Schools need to balance this approach with the knowledge that pupils who are already proficient in early mathematics can be negatively impacted if they are expected to go over old content, such as counting and basic shape knowledge.[footnote 65]

Maths facts, vocabulary and symbols at the start of a sequence

Pupils also need to know the core concepts, formulae and rules to draw on in topics such as algebra, geometry, statistics and calculus. Pupils who lack knowledge of concepts that they would normally have learned in previous key stages can benefit from additional topic-specific instruction.[footnote 66] This also has implications when making assumptions about pupils' general knowledge. For example, the concept of 'random phenomena' that would develop from pupils' exposure to games involving throwing dice can easily be missing from the schema for pupils who are learning about probability.[footnote 67] Core concepts should build seamlessly on knowledge acquired in previous phases. For example, a younger pupil's knowledge of the concept of 'balance' and the way this concept is connected to the equals sign will help them when they encounter linear equations in key stage 3.[footnote 68]

Case studies of curriculums for teaching algebra in countries where pupils do well also show that the conceptual building blocks of algebraic thinking are systematically planned into the earliest of curriculum stages. 'Variables, equations, equation solving, and function sense are permeated into the arithmetic analysis of quantitative relationships'.[footnote 69] This approach enables pupils to learn the concept of variables as well as the standard convention for using 'x' to represent an unknown variable when they are around 10 years old. They can then be taught, and apply, further codes, rules and principles of simple equations soon after. This approach shows that progression from arithmetic to algebra should be considered carefully by ensuring that pupils have the codes for number (maths facts, symbols, vocabulary) in place as a pre-requisite for moving on to a new topic or domain.[footnote 70]

Based on the above, high-quality maths education may have the following features

- Teachers engineer the best possible start for pupils by closing the school-entry gap in knowledge of the early mathematical code: facts, concepts, vocabulary and symbols.
- Pupils are taught core facts, formulae and concepts that are useful now and in the next stage of education.
- Teachers help pupils develop their automatic recall of core declarative knowledge, rather than rely on derivation, guesswork or casting around for clues.

Curriculum sequencing: procedural knowledge

Planned obsolescence of early methods

Ideally, pupils gradually cease to depend on some methods of counting and calculating, and associated resources, that they were taught earlier on. This is because reliance on some early counting and calculation methods, in the absence of learning valuable number facts, can hinder later progress.[footnote 71]

Pupils can be helped with simple everyday objects and semi-concrete representations, such as Numicon, but the aim should be that pupils move to working with symbols and abstract representations.[footnote 72] The use of manipulatives, for example, does not always guarantee that a pupil will understand[footnote 73] and their use may distract pupils from thinking about content to be learned.[footnote 74] A method of calculation that relies on derivation may be useful in the short term and as a bridge to formal methods of written calculation that require pupils to accurately recall number bonds.[footnote 75] In the absence of learning this core knowledge, pupils may rely too much on estimation and looking around for clues, or they may develop the habits of guessing and copying.[footnote 76]

In contrast, a visually simple counting frame (such as a soroban, commonly used in countries where pupils experience early success) is a resource that represents an efficient and powerful early method of calculation. The method associated with this resource, once the pupil has been taught how to use it, consistently presents accurate connections of number that can be learned and then later recalled as number sequences, rules and bonds. Giving young pupils an efficient, less distracting method of calculation that is not associated with other familiar activities (such as toys used for social play) helps them to see past the methods and any associated resources to new connections of number.[footnote 77] In the case of a simple counting frame, children no longer need it once they have learned to recall the number bonds, sequences, patterns and rules automatically.[footnote 78] The maths facts that they have acquired because of familiarity with and use of a powerful method can then aid their ability in mental arithmetic.[footnote 79] It is not the resource itself but the fact that its use is associated with efficiency, accuracy and visual simplicity that is the most important feature of powerful early methods.

Methods for more complex measurements and calculations

The ideal pen and paper methods in the 4 operations and for working with fractions are efficient, accurate and clear. The resulting neatness and logical approach helps to minimise the risk of pupils making accidental errors.[footnote 80] The appendices of the national curriculum in England for mathematics[footnote 81] give examples of the formal methods that pupils could use for the greatest likelihood of success in calculation as they progress through the curriculum.

Informal methods, some of which may involve physical resources, can be useful for revealing underlying principles and concepts.[footnote 82] However, teachers need to be cautious when considering curriculum approaches that are heavily weighted towards encouraging informal and self-generated methods. These approaches may purport to develop pupils' understanding, but the evidence shows that when pupils use a variety of informal procedures, it can inhibit understanding later on.[footnote 83]

Additional risks arise from mixing and matching a toolkit of informal and self-generated methods for working with larger numbers and more complex calculations as pupils progress through the curriculum. This increases the likelihood of pupils generating errors and structuring written records poorly, which may lead to confusion.[footnote 84]

Teachers should seek to balance developing pupils' understanding and its associated use of informal and diagrammatic methods with instruction in efficient methods that accurately and consistently reveal new patterns and connections of number. This is because the 2 aspects of understanding and computational proficiency reinforce and augment each other.[footnote 85] One way to achieve this is to plan to use informal methods for only a short amount of time, as a bridge to formal written methods. This would ensure that pupils have adequate opportunities to learn, rehearse and then use formal methods. The earlier learning and therefore increased use of core mathematical methods[footnote 86] also gives greater assurance to teachers that their pupils will be ready to use these methods within sequences of calculation and to solve more complex problems in their next phase of learning.

Methods for working algebraically

The message of quality over quantity of procedural knowledge also applies throughout key stages 3 and 4.

In algebra, pupils benefit from fewer but powerful representations and an iterative approach to sequencing the facts and procedures for working algebraically.[footnote 87] Abstract representations can be just as effective as contextualised representations.[footnote 88] The bar modelling method can be used as a bridge from arithmetic to early algebra. It is a useful interim method for abstracting arithmetic and algebraic expressions from word problems.[footnote 89] Teachers can even teach methods of evaluation of algebraic expressions and ways to set these out as a series of steps for pupils to learn by heart.[footnote 90] This contrasts with an approach of encouraging more informal, self-generated ways for pupils to solve linear equations. This may be self-limiting when pupils are faced with unconventional presentations of linear equations.

If pupils learn high-quality, useful and efficient procedural knowledge, they can then apply this to setting out and using formulae, from calculating areas and perimeters of different classes of polygon in key stage 2, to using the trigonometric formulae such as the cosine rule in key stage 4.

Based on the above, high-quality maths education may have the following features

- Teachers teach younger pupils non-distracting and accurate mathematical methods that encourage them to use recall over derivation.
- Teachers plan to teach older pupils efficient, systematic and accurate mathematical methods that they can use for more complex calculations and in their next stage of learning.
- Teachers help pupils to use these methods to see new connections of number, geometry and time.
- Teachers encourage pupils to use core mathematical methods rather than resort to guesswork, cast around for clues or use unstructured trial and error.

Curriculum sequencing: conditional knowledge

The importance of a curricular approach

Analysis of proficient mathematicians' problem-solving shows that their thinking is highly organised. It draws on a well-connected knowledge base of facts, methods and strategies that have been used to solve problems with a similar deep structure before.[footnote 91] Successful problem-solving is therefore not just an activity but an outcome of successful learning of the facts and methods, and their useful combinations as strategies. Conversely, if a problem-solver does not have conditional knowledge, they are more likely to be distracted by the surface features of problems.[footnote 92]

This has implications for how problem-solving as an activity is implemented in classrooms where teachers expect pupils to learn how to problem-solve by problem-solving.

Teachers could use a curricular approach that better engineers success in problem-solving by teaching:

- the useful combinations of facts and methods
- how to recognise the problem types
- the deep structures that these strategies pair to[footnote 93]

Pupils need to be fluent with the relevant facts and methods before being expected to learn how to apply them to problem-solving conditions.[footnote 94]

Giving younger pupils the ability to understand word problems

In the primary phases, pupils' experience of problem-solving often involves solving word problems. The first barrier to overcome is language. Pupils therefore need to be proficient readers at the required level.[footnote 95]

However, even when pupils can read and understand the context, they need to look beyond the surface features of the problem towards the deep structure that signals the strategy to be used. Some pupils who are quick to learn new procedural knowledge, such as how to find equivalent fractions, might be given linked problem-solving by way of differentiation. These pupils are then able to intuit the connections between recently learned methods and certain types of (word) problems. When they learn the conditions for a method's wider use, this builds a bank of strategies over time. They are then able to use that bank of strategies to classify and solve varied problems presented in test papers. However, pupils who have not had this experience will struggle to make the right connections when presented with problem-solving and reasoning test papers.

Teachers should therefore ensure that more pupils experience success in solving word problems, by sequencing the teaching of strategies to 'convert' the deep structure of word problems into simple equations.[footnote 96]

Strategies for solving classes of problem

Given that expertise is largely domain-specific,[footnote 97] strategies for solving problems are topic-specific and can therefore be planned into the content sequence for that topic.[footnote 98] This helps to prevent gaps in problem-solving strategies from emerging and means that the planned curriculum becomes more inclusive.

This contrasts with the view of problem-solving as a generic skill that pupils can transfer to multiple topics and sub-domains.[footnote 99] Applying generic strategies to find examples, look for relationships or weigh up features could yield accurate and inaccurate information. However, pupils who are struggling might not know which information to choose to use.

Pupils who are already proficient may experience success through using a generic process that involves more weighing up, sifting through and trial and error processing of information. Pupils who lack proficiency, on the other hand, experience frustration and learn less under these conditions.[footnote 100] Novices benefit from being given the ability to recognise the deep structure of a problem and to be able swiftly deploy a suitable strategy.[footnote 101] Pupils can then develop further conceptual understanding through applying procedures to classes of problem.[footnote 102]

Curriculum planning should seek to identify and sequence the most useful combinations of facts and methods for solving sub-classes of problems, as well as the features of conditions that these strategies would be useful for.

Based on the above, high-quality maths education may have the following features

- Teachers teach useful, topic-specific strategies to pupils, as well as how to match them to types of problem.

- Pupils are confident using linked facts and methods that are the building blocks of strategies, before strategies are taught.
- Teachers encourage pupils to use core, systematic strategies rather than resorting to guesswork or unstructured trial and error.

Curriculum sequencing: meeting pupils' needs

Planning for what pupils will be thinking about

Close examination of lesson planning and teachers' thoughts about lesson planning in education systems where pupils do well reveal an intense focus on underlying knowledge structures and connections rather than the surface coherence of activities and teaching. This means that teachers are planning for what pupils will be thinking about or with, not what they will be 'doing'.

Teachers create these opportunities for detailed, content-focused planning from knowledge of pupils' prior learning.[footnote 103] In Reception Year, activities might have a similar feel. But the intentional, and not the developmental, approach is still more likely to lead to children becoming proficient.[footnote 104] Teachers should be cautious about giving pupils ownership over their own path of progression through the curriculum. This is not just because of the influence of prior knowledge on progression through the curriculum, but because pupils might not know enough about future progression in mathematics to make the best choices now.[footnote 105] For example, deciding when and how much to rehearse basic calculations may inadvertently curtail later chances of success if pupils feel that immediate success and accuracy are the best signals to move on.

Further, the option of problem-solving as part of task differentiation does not guarantee that all pupils will learn problem-solving strategies. Leaders and teachers should ideally view learning of all core content, including the links between content, as an entitlement and therefore a pre-planned pathway for all pupils.

Balancing new learning and rehearsal of learning

A moment of understanding does not guarantee long-term learning. Pupils benefit from studying worked examples in addition to practising solving similar types of problems.[footnote 106]

Therefore, teachers need to balance introducing new content with pupils' need to spend time revisiting content.[footnote 107] There should be space within the curriculum for planned consolidation. Pupils should not be rushed through content.

This is easier if the mathematics curriculum focuses on core content early and leaders prioritise and value consolidation. Minimising off-task behaviour may also help to maximise the amount of time available for retrieval, rehearsal and consolidation of learning. Pupils who do well tend to have spent more time on the subject.[footnote 108]

Equity

Teachers and leaders should try to strike a balance between curricular approaches that enable pupils to keep up with their peers and reactive approaches that identify, help and support pupils after they have fallen behind. These reactive approaches are more likely to rely on assessment, diagnoses, personalisation and interventions.

In the English mathematics education system, emphases on reactive approaches are associated with a wide attainment spread and a long tail of under-achievement. Almost 180,000 students had to re-sit GCSE mathematics in 2019. Of these, only 22.3% achieved a standard pass (grade 4) or above. [footnote 109]

In East Asian classrooms, there appears to be little differentiation.[footnote 110] It might be assumed that this is the result of a pedagogical decision to keep pupils learning and doing the same thing. Teachers may worry that high attainers are being held back or that pupils with special educational needs and/or disabilities (SEND) are not being given enough support. However, in countries like Singapore, all groups of pupils do well. Fifty-one per cent of Singaporean pupils met the advanced international benchmark versus just 11% of English pupils. For the intermediate benchmark, described as the ability to 'apply basic mathematical knowledge in a variety of situations', only 8% of Singaporean children did not meet this standard, compared with 31% of English children.[footnote 111] The reason for this success is because a powerful curriculum and plenty of opportunities to engage in purposeful, intelligent practice lead to better outcomes for pupils.[footnote 112]

Leaders could consider this strategy as a way to promote proficiency in the subject, where pupils stay together not because higher attainers are being held back, but because lower attainers can 'keep up'.

Inclusivity

Pupils with SEND benefit hugely from explicit, systematic instruction and systematic rehearsal of declarative and procedural knowledge.[footnote 113] The benefits of these approaches extend beyond enhanced academic attainment and proficiency. The relationship between cognitive ability and academic attainment, including in numeracy, is in fact bidirectional.[footnote 114] Therefore, educational outcomes for pupils with SEND are likely to improve if teachers use systematic instruction and rehearsal to help pupils learn planned content.[footnote 115]

This approach is particularly useful for pupils with moderate learning difficulties who have slower cognitive processing speed.[footnote 116] Systematic approaches increase the amount of content considered per unit of time. These approaches are also highly beneficial in enhancing the progress, attainment and self-esteem of disadvantaged pupils.[footnote 117] Systematic curricular approaches give pupils with SEND and disadvantaged pupils a better chance of success, of keeping up and therefore of feeling included.

Playing to pupils' strengths: the powerful declarative memory systems of pupils with autism

Many pupils with autism have 'normal to above average algorithmic thinking ability' but can struggle with reasoning and problem-solving because of:

- language processing deficits
- difficulties in classifying problems by type
- lack of knowledge of strategies
- the use of 'inefficient and overly complex procedures' for calculation[footnote 118]

Teachers can fill these gaps in knowledge with systematic curriculums, teaching approaches and rehearsal. For example, teaching efficient algorithms to pupils with autism speeds up their calculations. They then have more time to learn strategies for solving classes of problem.

However, research also shows that the unique organisation and powerful declarative memory systems of many people with autism help them study, and develop proficiency in, the subject.[footnote 119] Potentially, a powerful declarative memory system can take on a compensatory role; thus many pupils with autism might benefit from a deliberate focus on memorisation of core facts and methods.

Leaders should therefore consider ways to give autistic pupils more time to study core content so that they can close gaps in learning through deliberate memorisation. Leaders should also make sure pupils' lesson time is used efficiently and effectively.

Based on the above, high-quality maths education may have the following features

- New content draws on and makes links with the content that pupils have previously acquired.
- Curriculum progression is by intelligent design rather than by choice or chance.
- Rehearsal sequences align with curriculum sequences.
- Pupils who are more likely to struggle or who are at risk of falling behind are given more time to complete tasks, rather than different tasks or curriculums, so that they can commit core facts and methods to long-term memory.

Pedagogy: new learning

Summary

In this section, we discuss the instructional needs of pupils as they progress through the curriculum. When pupils are at the start of school life or starting a new sequence of learning, they need more instruction than pupils who are already competent in that topic area. Throughout sequences of learning, pupils benefit from teaching that is systematic and clear. Pupils can also develop further understanding when sets of exercises are curated to present new and useful number connections, as pupils rehearse recently taught content.

The novice needs more instruction, not less

'Novice learners' of new mathematics content need systematic instructional approaches similar to those used to teach early reading and writing. Teachers need to ensure daily dedicated time for teaching and practising component parts.[footnote 120] Like the 'code' for language, it is useful to think about early mathematical content as also being a 'code'. Not all pupils will 'crack', discover or invent this code for themselves. An approach that comes closest to guaranteeing foundational success in mathematics is one that acknowledges that:

> To most effectively develop more comprehensive and abstract thinking about mathematics, children often need more than their natural, spontaneous learning.[footnote 121]

An approach like this should incorporate extra elements of explicit, systematic instruction. This will help to close the school entry gap in knowledge. It will also give more pupils the foundations for mathematical success,[footnote 122] as well as greater self-esteem.[footnote 123]

Use of intelligent variation in sets of exercises

There is a difference between content that pupils have recently learned and content that develops further in their minds through practice. Both can be planned for. Variation within sets of exercises can help pupils to learn:

- the ranges and boundaries of strategy applicability
- important patterns and rules
- connections between varying problems[footnote 124]
- pattern-seeking habits
- how to focus
- logical and systematic approaches to solving problems[footnote 125]

Leaders need to make sure they curate and control this approach. This is evident in the systematic use of variation in collections of tasks given to pupils in China, Hong Kong and Taiwan.[footnote 126]

Systematic instructional approaches also work well for all ages and stages

Proficient mathematicians are able to demonstrate success in problem-solving lessons.[footnote 127] However, it is easy then to assume that the activity that demonstrates a proficient mathematician's ability to problem-solve is the ideal means of acquiring proficiency.[footnote 128] Learning through participating in similarly open-ended problem-solving activities might be enjoyable for both teachers and pupils,[footnote 129] but it does not necessarily lead to improved results.[footnote 130] The adult in the room is an important mediator of pupils' success.

Without the adult, even where content and sequence of content might be ideal, the learning of at-risk groups of pupils is compromised.[footnote 131] Evidence shows that pupils can learn from worked examples,[footnote 132] particularly if teachers help pupils to make sense of worked examples.[footnote 133] Questioning, as long as teachers take care with language and timing, can also aid instruction.[footnote 134] Teaching pupils how to construct and use visual representations can help pupils to convert information presented in a problem into symbolic equations.[footnote 135]

Systematic instruction might also offer up benefits beyond enhanced learning of facts, methods and strategies because pupils who are more successful develop better learning behaviours.[footnote 136]

Based on the above, high-quality maths education may have the following features

- Teachers remember that it is not possible for pupils to develop proficiency by emulating expertise, but by emulating the journey to expertise.
- Systematic instructional approaches to engineer success in learning are incorporated into all stages and phases.
- Teachers aim to impart core content in alignment with the detail and sequence of the planned curriculum.
- Teachers help pupils to avoid relying on guesswork or unstructured trial and error.

Pedagogy: consolidation of learning

Summary

As pupils progress through the curriculum, they need regular opportunities to rehearse and apply the important facts, concepts, methods and strategies that they have learned. When designing sequences of rehearsal, teachers need to consider both the quality and the quantity of practice that pupils need to develop their understanding and to make core content firm and precise in the mind. Practice needs to go beyond immediate accuracy and understanding. Sequences of rehearsal should help to prevent pupils forgetting content over time.

Categories of rehearsal/consolidation of learning

Content category	Type 1 practice	Type 2 practice
Declarative	Fact retrieval (recall)	Explaining relationships between facts (derivation and parsing of number)
Procedural	Method rehearsal (exercises)	Explaining principles, proving conceptual understanding (such as, use of informal methods, creating bar models and interpreting context)
Conditional	Strategies rehearsal (collections of problems with the same deep structure)	Describing relationships between the problem and choices of strategy (proof/reasoning)

Quantity

Some pupils are quick to grasp new content, while others might need more time to think, practise, recall and apply. Given that proficiency in mathematics requires pupils to attain a level of procedural fluency,[footnote 137] teachers should ensure that they give pupils adequate opportunities to practise. This is more likely to increase pupils' levels of procedural fluency.

Consolidation of learning transforms pupils' initial moments of success, realisation and understanding into long-term memories.[footnote 138] The younger the pupil and the lower the level of overall mathematical skills, the more time and the greater the number of repetitions needed to attain automaticity in facts and methods.[footnote 139] If a pupil's recall fails, therefore, it might be that they need more practice[footnote 140] rather than just repeated teaching.[footnote 141]

In the most successful systems of mathematics education, systematic rehearsal is given more time and focus than in England. Powerful teaching and learning in classrooms where pupils do well are supported by regular homework assignments that require pupils to systematically rehearse content at home.[footnote 142] Teachers in these systems can plan future sequences of learning confident that pupils' foundational knowledge is secure. Under these conditions, consolidation of learning is personalised by time taken to complete assignments where:

> every child will have had to attend to every word, every problem, and every exercise included in every textbook.[footnote 143]

In contrast, pupils in England spend less time on mathematics homework than pupils in high-performing countries.[footnote 144] The fact that extra rehearsal, particularly in core content, helps pupils attain automaticity in recall and use of facts and methods[footnote 145] may explain some of the increases in attainment following the introduction of the 'numeracy hour' into English primary schools.[footnote 146] Conversely, when lessons and therefore rehearsal opportunities are cut, attainment declines.[footnote 147]

Comparison of textbooks also reveals that the expected volume of calculations, exercises and collections of problems to be completed is higher in countries where pupils tend to do well.[footnote 148] The evidence points to the need for teachers to provide enough opportunities to practise taught facts, methods and strategies, as well as additional opportunities for overlearning.[footnote 149] Efficient pedagogies such as choral response, explicit timing and goal setting may help to increase the 'rate' of practice in lessons, if it is difficult to provide additional opportunities for overlearning.[footnote 150]

Quality

Textbook analysis can provide useful information about quality of rehearsal. In contrast to the ideal of systematic rehearsal aligned to sequences of learning:

> English primary textbooks tend instead to move around rapidly and to constantly recapitulate.[footnote 151]

Lack of coherence and evidence-informed features in textbooks given to the youngest pupils could potentially influence the likelihood of later SEND.[footnote 152] This may be compounded by the fact that textbooks are generally seen by teachers in England as a supplemental resource rather than a potential and valuable system of rehearsal.[footnote 153] This is different to how textbooks are used in countries where pupils do best, not just in terms of volume of questions, but also in terms of sequencing and alignment with the curriculum sequence.[footnote 154]

Textbooks are particularly important for low attainers[footnote 155] and they might also be useful for pupil and parent buy-in. Pupils know they need to concentrate in the lesson to be able to complete the homework and they know they need to complete the homework to understand the next lesson.[footnote 156] Parents can also easily check their child's progress.[footnote 157]

Systematic rehearsal does not always require textbooks, pencil and paper. For younger pupils, rehearsal of number bonds and sequences can draw on a canon of games and songs involving dice, dominoes and counting sequences.[footnote 158] Computing technology can also help pupils acquire number facts by providing them with enough repetitions and direct feedback in ways that they enjoy.[footnote 159] Pupils also experience more progress and enjoyment of computer maths games when core content is introduced as separate learning components that are systematically followed by 'mini-games' than if content is entirely subsumed into the gaming conditions.[footnote 160] This approach can also be a useful intervention for pupils with SEND.[footnote 161] However, teachers should take care with this approach because not all children make the same progress when learning with computers.[footnote 162]

Tasks that are content-focused and achievable

In mathematics, studies suggest that long-term retrieval of core content should be a focus of teachers' and leaders' planning.[footnote 163] This means teachers should set pupils tasks that focus on rehearsal of facts, methods and strategies in addition to tasks that develop pupils' understanding.

Activity observation might show that pupils are engaged with and enjoying an activity, but if pupils are spending large amounts of time making choices, working out what to do or setting out, such as when physical apparatus is involved, their attention and learning can be compromised.[footnote 164] For example, drawing, measuring and comparing the angles in a polygon to find out and then learn the formula for the sum of angles means that pupils think about the formula in the last few minutes of the lesson. Even imagery can be distracting: textbooks in countries where pupils do well have fewer non-content related and distracting illustrations, pictures and cartoons.[footnote 165]

Pupils are more likely to engage in disruptive behaviours if they are expected to complete tasks that they have not mastered the component parts of yet. They are more likely to stay on task and be motivated if tasks are achievable.[footnote 166] In turn, sustained completion of tasks helps pupils to improve their ability to focus.[footnote 167] It is better for pupils to initially learn and rehearse content as component parts before learning the conditions for its use within a composite skill.[footnote 168] This has implications for 'challenge' because pupils tend to resort to using the methods they have most facility with, rather than those that are most valid and that have been recently taught, when faced with unfamiliar or demanding tasks. For example, pupils tackling tricky arithmetic problems will default to addition,[footnote 169] pupils who are new to working with algebra will default to arithmetic methods or trial and error,[footnote 170] and new learners of calculus will fall back on familiar concepts that are visually similar, but unrelated to the question.[footnote 171]

When pupils are ready to solve problems, they need to be able to hold a line of thought and to concentrate.[footnote 172] Background noise and general chit-chat have been shown to negatively affect pupils' ability to understand what the teacher is saying, to maintain appropriate behaviours and to concentrate. The children who are most affected are those who are under 13 and children with SEND.[footnote 173] Studies have also shown that the ideal environment for periods of independent work is one that is not just quiet but is in fact near silent.[footnote 174] That is not to say that all rehearsal experiences should be silent. Group work can aid pupils' development of explanations, providing it is tightly managed.[footnote 175] However, there are limits to its impact on learning as it does not always improve attainment and is difficult to implement.[footnote 176] Teachers should balance opportunities for discussion with pupils' needs for quiet periods of time to think.

Scaffolds as aids, not crutches

Teachers need to give careful consideration to how they use scaffolds, frames, physical apparatus and alternative information sources for pupils identified as needing extra support. There is a distinction to be made between using physical apparatus to reveal useful information[footnote 177] and its habitual use as an outsourced memory.

Reliance and subsequent dependence on manipulatives and associated aids can hinder progression through the curriculum.[footnote 178] The implications are that teachers need to give pupils enough time to consolidate learning and they need to plan for how pupils will move away from using the manipulative. This will help to avoid pupils relying on manipulatives to work around gaps in core knowledge that might become barriers to learning later.

Balancing rehearsal of proof and explanations with rehearsal of facts, methods and strategies

There are 2 'types' of practice:

- 'type 1' involves the rehearsal of core facts, methods and strategies that can be used to complete exercises and solve problems now and in the next stage of education
- 'type 2' includes explaining, justifying and proving concepts using informal and diagrammatic methods, parsing and derivation of number

Teachers need to create balance between these 2. It is helpful for pupils to replicate explanations and proof as a way of improving their own conceptual understanding of the 'why', but when it comes to learning how to find solutions to problems, practice of the methods of calculation themselves so that they can be recalled in the long term is likely to be a key to proficiency,[footnote 179] particularly for pupils who are identified as being more likely to struggle.[footnote 180] This gives greater assurance that pupils can use core knowledge of facts, efficient methods and useful strategies in the next stage of their education. Close inspection of curriculums in countries where pupils do well shows that systematic rehearsal emphasises learning and applying core facts and methods alongside, rather than after, the development of conceptual understanding.[footnote 181]

Based on the above, high-quality maths education may have the following features

Educators plan to give pupils opportunities to consolidate learning that:

- go beyond immediately answering questions correctly
- involve overlearning
- align with the detail and sequence of the curriculum
- are free of distraction and disruption
- strike a balance between type 1 and type 2 practices
- avoid creating a reliance on outsourced memory aids or physical resources
- help pupils to avoid relying on guesswork, casting around for clues or the use of unstructured trial and error

Assessment

Summary

Assessment during the learning journey is most useful when it focuses on the component knowledge that pupils have learned. This approach aids pupils' confidence and makes it easier to analyse and respond to gaps in learning. In mathematics, pupils benefit from timed practice of knowledge that should be easily recalled, such as maths facts. The timing element gives assurance that pupils are not reliant on derivation.

Frequent low-stakes testing helps to prepare pupils for the final performance

Summative assessments of learning need to provide easily comparable information to all stakeholders, including parents and the pupils themselves, on a regular basis.[footnote 182] Module exams provide short-term goals and a sense of achievement, but they can promote a 'just in time' approach to learning that means that knowledge is jettisoned soon after tests are taken. End-of-course examinations give greater assurance that the learning of content is long term.[footnote 183] This suggests that a mixture of approaches is best: regular tests of content recently taught and learned and an objective, fair and accurate summative assessment at the end of the year or course.

Leaders should, however, avoid conflating the 2 concepts with frequent use of summative tests, such as past papers. These can cause lower attaining pupils and pupils with SEND to be regularly reminded about what they do not know and cannot do (which may inculcate guesswork, misconception rehearsal or avoidance tactics). Over time, pupils who do not experience success can then become demotivated, which may negatively impact on their chances of attaining a pass when re-taking courses between 16 and 18.[footnote 184]

However, it is not the tests but lack of proficiency that causes this performance anxiety. Lack of proficiency can also be compounded with the use of 'realistic' settings of story problems that present a language barrier for disadvantaged children.[footnote 185] Teachers can ensure that pupils come to see tests and testing as moments to shine by adopting the principles underpinning the causal pathway to motivation and enjoyment in the subject. This can be achieved by seeking to engineer proficiency and initial success in the subject.

When pupils obtain levels of proficiency, they look forward to and enjoy tests.[footnote 186] Competitive maths games are, for example, more effective for learning and retention than non-competitive games.[footnote 187] The goals of trying to achieve a personal best and doing well compared to the average mediate later attainment.[footnote 188] Therefore, in addition to ensuring pupils are well prepared for tests, leaders should ensure that benchmarks for success are understandable.

Frequent, low-stakes testing of taught content can help prepare pupils for summative tests by providing memory-enhancing opportunities to recall and apply taught content.[footnote 189] Low-stakes testing also works well when tests of component parts, such as mathematics facts, are timed.[footnote 190] If teachers give honest feedback, pupils' interest and sense of self-efficacy also increases.[footnote 191] Teachers can also set benchmarks for mastery of facts and methods so that they can be assured that pupils are recalling rather than guessing or deriving.[footnote 192] Tests should therefore be aligned closely with curriculum sequences because generic tests are not able to give this feedback.[footnote 193]

Based on the above, high-quality maths education may have the following features

- Pupils are well prepared for assessments through having learned all the facts, methods and strategies that are likely to be tested.
- Teachers plan frequent, low-stakes testing to help pupils to remember content.
- Lessons incorporate timed testing to help pupils learn maths facts to automaticity.

Systems at the school level

Summary

Leaders can support pupils' progression through the mathematics curriculum by ensuring that pupils' bookwork is of a high quality. This is important because when pupils' calculations are systematic and orderly, they are better able to see the connections of number and to spot errors that can be corrected. Leaders can also plan to develop teachers' subject and subject-pedagogic knowledge through giving teachers opportunities to work with and learn from each other. This, for example, helps new teachers to see and adopt useful ways of explaining core concepts, methods and strategies to the pupils they teach.

Calculation and presentation

Accurate calculations and careful presentation give pupils the ability to spot important and interesting patterns of number, as well as errors that need to be corrected. Calculation methods and presentation rules are procedural knowledge that need to be taught and rehearsed to automaticity. Some pupils might naturally develop 'neatness' and subsequent accuracy, but teaching and rehearsing this procedural knowledge gives greater assurance that more pupils will be able to see errors and spot patterns of number, as well experience a sense of accomplishment.

That is not to say that more 'messy' experimental workings should never be allowed. However, teachers can help to engineer calculation and presentation success by balancing experimental approaches with opportunities to learn how to be systematic, logical and accurate when applying taught facts, methods and strategies.

Proactive professional development: the planned and purposeful pathway to expertise

The need for firm foundations applies to all novices, including novice mathematics teachers. Regular observations are often viewed as a main driver of professional development, where teachers are given feedback on aspects to improve. Use of the teacher standards for these reactive approaches prioritises on-the-ground development of pedagogical and subject-pedagogical expertise, highlighting features that are absent or in need of correcting along the way.

It may be tempting for leaders to focus on teacher-to-student relationships as an indicator of high-quality teaching and learning. However, analysis of pupils' attainment and attitudes suggests that a focus on pupils' effort and interest in the subject may matter more.[footnote 194] Given that initial teacher training can be variable in terms of pedagogical, subject-pedagogical and subject-specific knowledge,[footnote 195] we cannot assume that all novice mathematics teachers will possess all the tools they need to make the most successful start.

Leaders could consider incorporating more proactive approaches that close gaps and allow novice teachers to adopt and improve expert teaching methods, rather than develop their own aspects of effective mathematics teaching from scratch.[footnote 196] Such approaches could include:

- regular opportunities to observe and be mentored by experienced and successful teachers of mathematics
- provision of sequenced schemes of learning, matching textbooks and teacher notes to aid explanations and help the novice teacher to bring the subject to life[footnote 197]
- systematic plans to build these models of instruction and rehearsal over time so that future generations of teachers can benefit
- collaborative planning with more experienced and successful teachers of mathematics

Japanese lesson study is an example of a systematic approach to sharing subject-pedagogical knowledge that builds and shares subject-pedagogical knowledge at organisational, local and national scales.[footnote 198] The fact that lesson study is a system should also alert teachers and leaders to the dangers of adopting 'surface features' and not systems. This may also explain why attempts to install (the surface features of) 'lesson study' as a curricular or pedagogical intervention leads to somewhat less convincing results.[footnote 199]

Teachers should also seek to renew and improve their subject knowledge, even if they are teaching foundational concepts.[footnote 200] For example, teachers of primary age groups gain when they know the foundational principles that pupils can learn to help them with later algebra lessons.[footnote 201]

Based on the above, high-quality maths education may have the following features

- School-wide approaches to calculation and presentation in pupils' books.

- School-wide approaches to providing time and resources for teachers to develop subject knowledge and to learn valuable ways of teaching from each other.

Conclusion

Throughout the review, the theme of engineering success, underpinned by systems thinking, predominates. These approaches seek to transform an offer of content into more of a guarantee that content can and will be learned. The outcomes of this systems thinking are the observed features and approaches of successful mathematics education:

- detailed codification and sequencing of the facts, methods and strategies that pupils will acquire
- instructional coherence and aligned rehearsal that increase the chances of understanding and remembering while minimising the need for guesswork or trial and error

Within these powerful mathematics education systems, the textbooks, teacher guides and workbooks are seen as a vital part of the infrastructure for efficiently transmitting subject knowledge and subject-pedagogical knowledge to new generations of pupils and teachers. This signals a need for teachers and leaders to avoid installing features and approaches in the absence of the 'infrastructure' underpinning their efficacy. It is also likely that the features that tend not to be observed or selected, such as the less glamorous quality and quantity of practice, are also integral to the overall success of novice mathematicians.

Quality and quantity of practice is a vital key that unlocks the development of dual tracks of conceptual understanding and procedural fluency. Further, in observing pupils' relative expertise and proficiency, such as in a problem-solving lesson, teachers and leaders should be mindful of the journey that pupils took to achieve problem-solving proficiency. This journey will have involved more than the features and activities of the lessons that proficient mathematicians are taking part in at the time. Variation in the quality of mathematics education in England is likely to be the result of the absence of systems and systems thinking, as well as possible gaps in content, instruction, rehearsal, assessment and the plans for their evolution over time.

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